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LIMIT THEORY FOR BIVARIATE CENTRAL AND BIVARIATE INTERMEDIATE DUAL GENERALIZED ORDER STATISTICS

BY

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Abstract. Burkschat et al. (2003) have introduced the concept of dual generalized order statistics (dgos) to unify several models that produce descendingly ordered random variables (rv's) like reversed order statistics, lower k-records and lower Pfeifer records. In this paper we derive the limit distribution functions (df's) of bivariate central and bivariate intermediate m-dgos. It is revealed that the convergence of the marginals of the m-dgos implies the convergence of the joint df. Moreover, we derive the conditions under which the asymptotic independence between the two marginals occurs.

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1. INTRODUCTION

Generalized order statistics (gos) have been introduced in Kamps [8] as a unification of several models of ascendingly ordered rv's. The use of the gos model has been steadily growing along the years. This is due to the fact that this model includes important well-known submodels that have been separately treated in statistical literature. Theoretically, many of the models of ordered rv's are contained in the gos model, such as ordinary order statistics (oos), order statistics with non-integral sample size, sequential order statistics (sos), record values, Pfeifer's record model and progressive type II censored order statistics (pos). These models can be applied in reliability theory. For instance, the sos model is an extension of the oos model and serves as a model describing certain dependencies or interactions among the system components caused by failures of components, and the pos model is an important method of obtaining data in lifetime tests. Live units removed early on can be readily used in other tests, thereby saving cost to the experimenter. Random variables that are decreasingly ordered cannot be integrated into the framework of gos. Therefore, Burkschat et al. [4] have introduced the concept of dual generalized order statistics (dgos) to unify several models that produce descendingly ordered rv's like reversed order statistics and lower records models. Uniform dgos $U^*_{d;r:n},\ r=1,2,\ldots,n$, are defined by their probability density function (pdf)

$$f^{U^*_{d;1:n},U^*_{d;2:n},\ldots,U^*_{d;n:n}}(u_1,u_2,\ldots,u_n)=\Big(\prod_{j=1}^{n}\gamma_j\Big)\Big(\prod_{j=1}^{n-1}u_j^{\gamma_j-\gamma_{j+1}-1}\Big)u_n^{\gamma_n-1},$$

where $1\geqslant u_1\geqslant\ldots\geqslant u_n>0$. The parameters $\gamma_1,\gamma_2,\ldots,\gamma_n$ are defined by $\gamma_n=k>0$ and $\gamma_r=k+n-r+M_r,\ r=1,2,\ldots,n-1$, where $M_r=\sum_{j=r}^{n-1}m_j$ and $\tilde{m}=(m_1,\,m_2,\ldots,m_{n-1})\in\Re^{n-1}$. The dual rv based on a df F is defined by the quantile transformation $X_d(r,n,\tilde{m},k)\overset{d}{=}F^{\leftarrow}(U^*_{d;r:n}),\ r=1,2,\ldots,n$, where $F^{\leftarrow}$ denotes the quantile function of F, i.e., $F^{\leftarrow}(y)=\inf\{x:F(x)\geqslant y\}$ ($\overset{d}{=}$ means identical distribution). The connections between m-gos and m-dgos are also established in Cramer [6] and Burkschat et al. [4]. Nasri-Roudsari [10] (see also Barakat [1]) has derived the marginal df of the rth m-gos, $m\neq-1$, in the form $\Phi^{(m,k)}_{r:n}(x)=I_{G_m(x)}(r,N-r+1)$, where $G_m(x)=1-\big(1-F(x)\big)^{m+1}=1-\bar{F}^{m+1}(x),\ I_x(a,b)=\frac{1}{\beta(a,b)}\int_0^x t^{a-1}(1-t)^{b-1}dt$ denotes the incomplete beta ratio function, and $N=\frac{k}{m+1}+n-1$. By using the well-known relation $I_x(a,b)=1-I_{\overline{x}}(b,a)$, where $\overline{x}=1-x$, and by putting $T_m(x)=F^{m+1}(x)$, the marginal df of the rth m-dgos, $m\neq-1$, is given by $\Phi^{d(m,k)}_{r:n}(x)=I_{T_m(x)}(N-r+1,r)$. Moreover, using the results of Burkschat et al. [4], we can write explicitly the joint pdf's of the rth and sth m-dgos, $m\neq-1$, $1\leqslant r<s\leqslant n$, in the form

(1.1)
$$\begin{aligned}f^{d(m,k)}_{r,s:n}(x,y)=\frac{C_{s-1,n}}{\Gamma(r)\Gamma(s-r)}F^m(x)\Big(g_m\big(\overline{F}(y)\big)-g_m\big(\overline{F}(x)\big)\Big)^{s-r-1}g_m^{r-1}\big(\overline{F}(x)\big)\\ \times F^{\gamma_s-1}(y)f(x)f(y),\qquad -\infty<y<x<\infty,\end{aligned}$$

where $C_{s-1,n}=\prod_{i=1}^{s}\gamma_i$, $s=1,2,\ldots,n$, and $g_m(x)=\frac{1}{m+1}[1-\overline{x}^{m+1}]$. In the present paper we reveal the asymptotic structural dependence between the members of dgos with variable ranks. The limit joint df of the m-dgos $X_d(r,n,m,k)$ and $X_d(s,n,m,k)$ for $m\neq-1$ is derived in the following two cases:

(1) Central case, where $r,s\to\infty$ and $r/N\to\lambda_1, s/N\to\lambda_2$, for $0<\lambda_1<\lambda_2<1$, as $N\to\infty$ (or, equivalently, as $n\to\infty$). A remarkable example of the central oos is the pth sample quantile, where $r_n=[np], 0<p<1$, and $[x]$ denotes the largest integer not exceeding x (see [7]).

(2) Intermediate case, where $r,s\to\infty$ and $r/N, s/N\to 0$ as $N\to\infty$ (or, equivalently, as $n\to\infty$). The intermediate oos have many applications, e.g., in the theory of statistics they can be used to estimate probabilities of future extreme observations and to estimate tail quantiles of the underlying distribution that are

extreme relative to the available sample size, see Pickands [11]. Many authors, e.g., Teugels [13] and Mason [9], have also found estimates that are based, in part, on intermediate order statistics.

The asymptotic bivariate df of the extreme m-dgos is derived in Barakat et al. [2]. Moreover, the asymptotic behavior for bivariate central and intermediate gos is investigated in Barakat et al. [3]. Everywhere in what follows the symbols $\underset{n}{\rightarrow}$ and $\overset{w}{\underset{n}{\rightarrow}}$ stand for convergence, as $n \to \infty$, and weak convergence, as $n \to \infty$, respectively.

2. LIMIT DISTRIBUTION FUNCTIONS OF THE BIVARIATE CENTRAL m-dgos

Consider a variable rank sequence $r = r_n \underset{n}{\rightarrow} \infty$ and $\sqrt{n}(r_n/n - \lambda) \underset{n}{\rightarrow} 0$, where $0 < \lambda < 1$. Smirnov [12] showed that if there exist normalizing constants $\tilde{\alpha}_n > 0$ and $\tilde{\beta}_n$ such that

$$\Phi_{r:n}^{d(0,1)}(\tilde{\alpha}_n x + \tilde{\beta}_n) = I_{F(\tilde{\alpha}_n x + \tilde{\beta}_n)}(n - r + 1, r) \overset{w}{\underset{n}{\rightarrow}} \Phi^{d(0,1)}(x; \lambda), \tag{2.1}$$

where $\Phi^{d(0,1)}(x; \lambda)$ is some nondegenerate df, then $\Phi^{d(0,1)}(x; \lambda)$ must have one and only one of the types $\mathcal{N}\big(W_{i,\beta}(x)\big)$, $i = 1, 2, 3, 4$, where $\mathcal{N}(\cdot)$ denotes the standard normal df, and

$$W_{1;\beta}(x) = \begin{cases} -\infty, & x \leqslant 0, \\ cx^\beta, & x > 0, \end{cases} \qquad W_{2;\beta}(x) = \begin{cases} -c|x|^\beta, & x \leqslant 0, \\ \infty, & x > 0, \end{cases}$$

$$W_{3;\beta}(x) = \begin{cases} -c_1|x|^\beta, & x \leqslant 0, \\ c_2 x^\beta, & x > 0, \end{cases} \qquad W_{4;\beta}(x) = W_4(x) = \begin{cases} -\infty, & x \leqslant -1, \\ 0, & -1 < x \leqslant 1, \\ \infty, & x > 1, \end{cases}$$

and $\beta, c, c_1, c_2 > 0$. In this case we say that F belongs to the λ-*normal domain of attraction* of the limit df $\Phi^{d(0,1)}(x; \lambda)$, written $F \in \mathcal{D}_\lambda\big(\Phi^{d(0,1)}(x; \lambda)\big)$. Moreover, (2.1) is satisfied with $\Phi^{d(0,1)}(x; \lambda) = \mathcal{N}\big(W_{i;\beta}(x)\big)$ for some $i \in \{1, 2, 3, 4\}$ if and only if

$$\sqrt{n}\frac{\lambda - \overline{F}(\tilde{\alpha}_n x + \tilde{\beta}_n)}{C_\lambda} \to W_{i,\beta}(x), \qquad \text{where } C_\lambda = \sqrt{\lambda(1 - \lambda)}.$$

It is worth mentioning that the condition $\sqrt{n}(r_n/n - \lambda) \underset{n}{\rightarrow} 0$ is necessary to have a unique limit law for any two ranks r, r' for which $\lim_{n\to\infty} r/n = \lim_{n\to\infty} r'/n$. The following lemma characterizes the possible limit laws of the df $\Phi_{r:n}^{d(m,k)}(x)$.

LEMMA 2.1. *Let* $r = r_n$ *be such that* $\sqrt{n}(r/n - \lambda) \underset{n}{\to} 0$, *where* $0 < \lambda < 1$. *Furthermore, let* $m_1 = m_2 = \ldots = m_{n-1} = m > -1$. *Then there exist normalizing constants* $\tilde{a}_n > 0$ *and* $\tilde{b}_n$ *for which*

$$\Phi_{r:n}^{d(m,k)}(\tilde{a}_n x + \tilde{b}_n) \xrightarrow[n]{w} \Phi^{d(m,k)}(x; \lambda), \tag{2.2}$$

where $\Phi^{d(m,k)}(x; \lambda)$ *is a nondegenerate df if and only if*

$$\sqrt{n}\, \frac{\lambda - \overline{T}_m(\tilde{a}_n x + \tilde{b}_n)}{C_\lambda} \underset{n}{\to} U(x),$$

where $\Phi^{d(m,k)}(x; \lambda) = \mathcal{N}\big(U(x)\big)$. *Moreover,* (2.2) *is satisfied for some nondegenerate df* $\Phi^{d(m,k)}(x; \lambda)$ *if and only if* $F \in \mathcal{D}_{\lambda(m)}\big(\mathcal{N}\big(W_{i;\beta}(x)\big)\big)$ *for some* $i \in \{1, 2, 3, 4\}$, *where* $\lambda(m) = 1 - \overline{\lambda}^{1/(m+1)}$ *and* $\overline{\lambda} = 1 - \lambda$. *In this case we have*

$$U(x) = \frac{C^*_{\lambda(m)}}{C^*_\lambda}(m+1)W_{i;\beta}(x), \quad \text{where } C^*_\lambda = \frac{C_\lambda}{\overline{\lambda}}.$$

P r o o f. The proof follows by using the same argument that is applied in the proof of Theorem 2.2 of Barakat [1] in the case of central m-gos. ∎

We assume in this section that $r = r_n$, $s = s_n \underset{n}{\to} \infty$ and $\sqrt{n}(r/n - \lambda_1) \underset{n}{\to} 0$, $\sqrt{n}(s/n - \lambda_2) \underset{n}{\to} 0$, where $0 < \lambda_1 < \lambda_2 < 1$. Moreover, we assume that there are suitable normalizing constants $\tilde{a}_n, \tilde{c}_n > 0$ and $\tilde{b}_n, \tilde{d}_n$ for which

$$\Phi_{r:n}^{d(m,k)}(\tilde{a}_n y + \tilde{b}_n) \xrightarrow[n]{w} \Phi^{d(m,k)}(y;\lambda_1) \quad \text{and} \quad \Phi_{s:n}^{d(m,k)}(\tilde{c}_n x + \tilde{d}_n) \xrightarrow[n]{w} \Phi^{d(m,k)}(x;\lambda_2),$$

where $\Phi^{d(m,k)}(y;\lambda_1)$ and $\Phi^{d(m,k)}(x;\lambda_2)$ are nondegenerate df's. Let $\Phi_{r,s:n}^{d(m,k)}(x, y)$ be the joint df's of the rth and sth m-dgos, $m \neq -1$. By (1.1) we get $\Phi_{r,s:n}^{d(m,k)}(x, y) = \Phi_{s:n}^{d(m,k)}(x)$, $x \leqslant y$, and

$$\Phi_{r,s:n}^{d(m,k)}(x, y) = \tag{2.3}$$
$$= C_n^* \int\limits_0^{F(y)} \int\limits_\eta^{F(x)} \xi^m \eta^{\gamma_s - 1}(1 - \xi^{m+1})^{r-1}(\xi^{m+1} - \eta^{m+1})^{s-r-1} d\xi d\eta, \quad x \geqslant y,$$

where

$$C_n^* = \frac{(m+1)^2 \Gamma(N+1)}{\Gamma(N - s + 1)(r-1)!(s-r-1)!}.$$

The following lemma concerning the asymptotic behavior of the df $\Phi_{r,s:n}^{d(m,k)}(x, y)$ is an essential tool in studying the limit df of the bivariate central m-dgos.

LEMMA 2.2. *Let* $\lambda_i = i/(N+1),\ \nu_i = 1-\lambda_i,\ \tau_i = \sqrt{\lambda_i\nu_i/(N+1)}$ *for* $i = r, s,\ 0 < R_{rs} = \sqrt{\lambda_r(1-\lambda_s)/\big(\lambda_s(1-\lambda_r)\big)} < 1$, *and let*

$$U_n^{d(1)}(y) = \frac{\lambda_r - \overline{T}_m(\tilde{y}_n)}{\tau_r}, \qquad U_n^{d(2)}(x) = \frac{\lambda_s - \overline{T}_m(\tilde{x}_n)}{\tau_s},$$

where $\tilde{x}_n = \tilde{a}_n x + \tilde{b}_n$ *and* $\tilde{y}_n = \tilde{c}_n y + \tilde{d}_n$. *Then*

$$\left|\Phi_{r,s:n}^{d(m,k)}(\tilde{x}_n,\tilde{y}_n) - \frac{1}{2\pi\sqrt{1-R_{rs}^2}}\int_{-\infty}^{U_n^{d(1)}(y)}\int_{-\infty}^{U_n^{d(2)}(x)} W_{r,s}(\xi,\eta)d\xi d\eta\right| \underset{n}{\rightarrow} 0$$

uniformly with respect to x *and* y, *where*

$$W_{r,s}(\xi,\eta) = \frac{1}{2\pi\sqrt{1-R_{rs}^2}}\exp\left(-\frac{\xi^2 - 2\xi\eta R_{rs} + \eta^2}{2(1-R_{rs}^2)}\right).$$

P r o o f. For given $\epsilon > 0$ choose T large enough to satisfy both of the inequalities $1/T^2 < \epsilon$ and $\mathcal{N}(-T) < \epsilon$. If $U_n^{d(1)}(y) \leqslant -T$, then, for sufficiently large n, we have $F^{m+1}(\tilde{y}_n) < \nu_r - \tau_r T < 1$. Therefore, after routine calculations, we can show that

$$\Phi_{r:n}^{d(m,k)}(\tilde{y}_n) = \frac{1}{\beta(N-r+1,r)}\int_0^{F^{m+1}(\tilde{y}_n)} \eta^{N-r}(1-\eta)^{r-1}d\eta$$
$$\leqslant \frac{1}{\beta(N-r+1,r)}\int_0^{\nu_r-\tau_r T} \eta^{N-r}(1-\eta)^{r-1}d\eta$$
$$\leqslant \frac{1}{\beta(N-r+1,r)}\int_0^1 \frac{(\eta-\nu_r)^2}{\tau_r^2 T^2}\eta^{N-r}(1-\eta)^{r-1}d\eta = \frac{N+1}{(N+2)T^2} < \frac{1}{T^2} < \epsilon.$$

Since $\Phi_{r,s:n}^{d(m,k)}(\tilde{x}_n,\tilde{y}_n) \leqslant \Phi_{r:n}^{d(m,k)}(\tilde{y}_n)$, it follows that $\Phi_{r,s:n}^{d(m,k)}(\tilde{x}_n,\tilde{y}_n) < \epsilon$. Similarly, if $U_n^{d(2)}(x) \leqslant -T$, we can prove that $\Phi_{r,s:n}^{d(m,k)}(\tilde{x}_n,\tilde{y}_n) \leqslant \Phi_{s:n}^{d(m,k)}(\tilde{x}_n) < \epsilon$. On the other hand, we have

$$\int_{-\infty}^{U_n^{d(1)}(y)}\int_{-\infty}^{U_n^{d(2)}(x)} W_{r,s}(\xi,\eta)d\xi d\eta \leqslant \min\Big(\mathcal{N}\big(U_n^{d(1)}(y)\big), \mathcal{N}\big(U_n^{d(2)}(x)\big)\Big) < \epsilon.$$

Therefore, if $U_n^{d(1)}(y) \leqslant -T$ or $U_n^{d(2)}(x) \leqslant -T$, we get

$$\Big|\Phi_{r,s:n}^{d(m,k)}(\tilde{x}_n,\tilde{y}_n) - \int_{-\infty}^{U_n^{d(1)}(y)}\int_{-\infty}^{U_n^{d(2)}(x)} W_{r,s}(\xi,\eta)d\xi d\eta\Big| < 2\epsilon.$$

Now, if $U_n^{d(1)}(y) \geqslant T$, then $F^{m+1}(\tilde{y}_n) \geqslant \nu_r + \tau_r T$. Therefore, we get

$$1 - \Phi_{r:n}^{d(m,k)}(\tilde{y}_n) \leqslant \frac{1}{\beta(N-r+1,r)} \int_{\nu_r+\tau_r T}^{1} \eta^{N-r}(1-\eta)^{r-1} d\eta$$

$$\leqslant \frac{1}{\beta(N-r+1,r)} \int_0^1 \frac{(\eta-\nu_r)^2}{\tau_r^2 T^2} \eta^{N-r}(1-\eta)^{r-1} d\eta = \frac{N+1}{(N+2)T^2} < \frac{1}{T^2} < \epsilon.$$

Thus, we also obtain

$$\Phi_{s:n}^{d(m,k)}(\tilde{x}_n) - \Phi_{r,s:n}^{d(m,k)}(\tilde{x}_n, \tilde{y}_n) < 1 - \Phi_{r:n}^{d(m,k)}(\tilde{y}_n) < \epsilon. \tag{2.4}$$

On the other hand, in view of our assumptions and Lemma 2.1, we get

(2.5)

$$\mathcal{N}\big(U_n^{d(2)}(x)\big) - \int_{-\infty}^{U_n^{d(1)}(y)} \int_{-\infty}^{U_n^{d(2)}(x)} W_{r,s}(\xi,\eta) d\xi d\eta = \int_{U_n^{d(1)}(y)}^{\infty} \int_{-\infty}^{U_n^{d(2)}(x)} W_{r,s}(\xi,\eta) d\xi d\eta$$

$$\leqslant \frac{1}{\sqrt{2\pi}} \int_{U_n^{d(1)}(y)}^{\infty} \exp\left(-\frac{\xi^2}{2}\right) d\xi \leqslant \frac{1}{\sqrt{2\pi}} \int_T^{\infty} \exp\left(\frac{-\xi^2}{2}\right) d\xi < \epsilon$$

for sufficiently large n, and

$$\Big|\Phi_{s:n}^{d(m,k)}(\tilde{x}_n) - \mathcal{N}\big(U_n^{d(2)}(x)\big)\Big| < \epsilon \tag{2.6}$$

for sufficiently large n. The inequalities (2.4)–(2.6) show that when $U_n^{d(1)}(y) \geqslant T$, we have

$$\Big|\Phi_{r,s:n}^{d(m,k)}(\tilde{x}_n, \tilde{y}_n) - \int_{-\infty}^{U_n^{d(1)}(y)} \int_{-\infty}^{U_n^{d(2)}(x)} W_{r,s}(\xi,\eta) d\xi d\eta\Big| < 3\epsilon.$$

Similarly, we can show that the last inequality holds for sufficiently large n if $U_n^{d(2)}(x) \geqslant T$. In order to complete the proof of the lemma, we have to consider the case $|U_n^{d(1)}(y)|, |U_n^{d(2)}(x)| < T$. First, we note that, in this case, for sufficiently large n, since $\overline{T}_m(\tilde{y}_n) \underset{n}{\rightarrow} \lambda_1 < \lambda_2 \underset{n}{\leftarrow} \overline{T}_m(\tilde{x}_n)$, we have $\tilde{x}_n \geqslant \tilde{y}_n$. Therefore, for sufficiently large n, $\Phi_{r,s:n}^{d(m,k)}(\tilde{x}_n, \tilde{y}_n)$ is given by (2.3). On the other hand, in this case we get $1 - F^{m+1}(\tilde{y}_n) < \lambda_r + \tau_r T \geqslant 0$ and $1 - F^{m+1}(\tilde{x}_n) < \lambda_s + \tau_s T \geqslant 0$. Therefore,

$$\Phi_{r,s:n}^{d(m,k)}(\tilde{x}_n, \tilde{y}_n) = \int_{1-F^{m+1}(\tilde{y}_n)}^{1} \int_{1-F^{m+1}(\tilde{x}_n)}^{w} \varphi_{r,s:n}^{d(m,k)}(w,z) dz dw = \tag{2.7}$$

$$\int_{1-F^{m+1}(\tilde{y}_n)}^{\lambda_r+\tau_r T} \int_{1-F^{m+1}(\tilde{x}_n)}^{\lambda_s+\tau_s T} \varphi_{r,s:n}^{d(m,k)}(w,z) dz dw + \int_{1-F^{m+1}(\tilde{y}_n)}^{\lambda_r+\tau_r T} \int_{\lambda_s+\tau_s T}^{w} \varphi_{r,s:n}^{d(m,k)}(w,z) dz dw$$

$$+ \int_{\lambda_r+\tau_r T}^{1} \int_{1-F^{m+1}(\tilde{x}_n)}^{w} \varphi_{r,s:n}^{d(m,k)}(w,z) dz dw,$$

where

$$\varphi^{d(m,k)}_{r,s:n}(w,z) = \frac{C^*_n}{(m+1)^2} z^{r-1}(1-w)^{N-s}(w-z)^{s-r-1}.$$

We shall separately consider each of the integrals in the summation (2.7). We have

$$\text{(2.8)} \quad \int\limits_{\lambda_r+\tau_r T}^{1} \int\limits_{1-F^{m+1}(\tilde{x}_n)}^{w} \varphi^{d(m,k)}_{r,s:n}(w,z)dzdw \leqslant \int\limits_{0}^{\lambda_r+\tau_r T} \int\limits_{1}^{w} \varphi^{d(m,k)}_{r,s:n}(w,z)dzdw$$

$$= \frac{C^*_n}{(m+1)^2} \int\limits_{0}^{\lambda_r+\tau_r T} \int\limits_{1}^{w} z^{r-1}(1-w)^{N-s}(w-z)^{s-r-1}dzdw$$

$$= \frac{\Gamma(N+1)}{\Gamma(N-r+1)\Gamma(r)} \int\limits_{0}^{\lambda_r+\tau_r T} w^{r-1}(1-w)^{N-r}dw < \frac{1}{T^2} < \epsilon,$$

(2.9)

$$\int\limits_{1-F^{m+1}(\tilde{y}_n)}^{\lambda_r+\tau_r T} \int\limits_{\lambda_s+\tau_s T}^{w} \varphi^{d(m,k)}_{r,s:n}(w,z)dzdw \leqslant \int\limits_{0}^{\lambda_s+\tau_s T} \int\limits_{w}^{\lambda_s+\tau_s T} \varphi^{d(m,k)}_{r,s:n}(w,z)dzdw$$

$$= \frac{\Gamma(N+1)}{\Gamma(N-s+1)(s-1)!} \int\limits_{0}^{\lambda_s+\tau_s T} z^{s-1}(1-z)^{N-s}dz < \frac{1}{T^2} < \epsilon,$$

and by using the transformation $z = \lambda_r - \xi\tau_r$, $w = \lambda_s - \eta\tau_s$, the third integral takes the form

$$\int\limits_{1-F^{m+1}(\tilde{y}_n)}^{\lambda_r+\tau_r T} \int\limits_{1-F^{m+1}(\tilde{x}_n)}^{\lambda_s+\tau_s T} \varphi^{d(m,k)}_{r,s:n}(w,z)dzdw = A_{r,s:n} \int\limits_{-T}^{U^{d(1)}_n(y)} \int\limits_{-T}^{U^{d(2)}_n(x)} g^{(d)}_{r,s:n}(\xi,\eta)d\eta d\xi,$$

where

$$A_{r,s:n} = \frac{\Gamma(N+1)\tau_r\tau_s\lambda_r^{r-1}\nu_s^{N-s}(\lambda_s-\lambda_r)^{s-r-1}}{\Gamma(N-s+1)(r-1)!(s-r-1)!}$$

and

$$g^{(d)}_{r,s:n}(\xi,\eta) = \left(1-\frac{\xi\tau_r}{\lambda_r}\right)^{r-1}\left(1-\frac{\eta\tau_s-\xi\tau_r}{\lambda_s-\lambda_r}\right)^{s-r-1}\left(1+\frac{\eta\tau_s}{\nu_s}\right)^{N-s}.$$

On the other hand, by Stirling's formula $\Gamma(M+1) = e^{-M}\sqrt{2\pi M}M^M\big(1+o(1)\big)$ as $M \to \infty$, we get

$$A_{r,s:n} = \frac{(N+1)^2\Gamma(N+1)\tau_r\tau_s\lambda_r^r\nu_s^{N-s}(\lambda_s-\lambda_r)^{s-r}}{\Gamma(N-s+1)r!(s-r)!}$$

$$= \frac{1+o(1)}{2\pi\sqrt{\frac{(N+1)(s-r)}{s(N-r)}}} = \frac{1+o(1)}{2\pi\sqrt{1-R^2_{rs}}}.$$

Also, it is easy to show that

(2.10)

$$\begin{aligned} g^{(d)}_{r,s:n}(\xi,\eta) &= \left(1-\frac{\xi\tau_r}{\lambda_r}\right)^{r}\left(1-\frac{\eta\tau_s-\xi\tau_r}{\lambda_s-\lambda_r}\right)^{s-r}\left(1+\frac{\eta\tau_s}{\nu_s}\right)^{N-s} \\ &\quad\times\left[\left(1-\frac{\xi\tau_r}{\lambda_r}\right)^{-1}\left(1-\frac{\eta\tau_s-\xi\tau_r}{\lambda_s-\lambda_r}\right)^{-1}\right] \\ &= \left(1-\frac{\xi\tau_r}{\lambda_r}\right)^{r}\left(1-\frac{\eta\tau_s-\xi\tau_r}{\lambda_s-\lambda_r}\right)^{s-r}\left(1+\frac{\eta\tau_s}{\nu_s}\right)^{N-s} \\ &\quad\times\left[\left(1+\frac{\xi\tau_r}{\lambda_r}(1+o(1))\right)\left(1+\frac{\eta\tau_s-\xi\tau_r}{\lambda_s-\lambda_r}(1+o(1))\right)\right] \\ &= \left(1-\frac{\xi\tau_r}{\lambda_r}\right)^{r}\left(1-\frac{\eta\tau_s-\xi\tau_r}{\lambda_s-\lambda_r}\right)^{s-r}\left(1+\frac{\eta\tau_s}{\nu_s}\right)^{N-s}\big(1+\rho_n(\xi,\eta)\big), \end{aligned}$$

where $\rho_n(\xi,\eta)\underset{n}{\rightarrow}0$ uniformly in any finite interval $(-T,T)$ of the values ξ and η. On the other hand, we have

$$\begin{aligned} r\ln\left(1-\frac{\xi\tau_r}{\lambda_r}\right) &= -r\left(\frac{\xi\tau_r}{\lambda_r}-\frac{\xi^2\tau_r^2}{2\lambda_r^2}+\frac{\xi^3\tau_r^3}{3\lambda_r^3}+\dots\right) \\ &= -\xi\tau_r(N+1)-\frac{\xi^2\nu_r}{2}+o\left(\frac{T^3}{\sqrt{r}}\right), \end{aligned} \tag{2.11}$$

$$\begin{aligned} (s-r)\ln&\left(1-\frac{\eta\tau_s-\xi\tau_r}{\lambda_s-\lambda_r}\right) \\ &= -(\eta\tau_s-\xi\tau_r)(N+1)-\frac{1}{2}\frac{(\eta\tau_s-\xi\tau_r)^2}{\lambda_s-\lambda_r}(N+1)+o\left(\frac{T^3}{\sqrt{s}}\right) \end{aligned} \tag{2.12}$$

and

$$(N-s)\ln\left(1-\frac{\eta\tau_s}{\nu_s}\right)=\eta\tau_s(N+1)-\frac{1}{2}\eta^2\lambda_s+o\left(\frac{\lambda_s^{3/2}T^3}{\sqrt{N}}\right). \tag{2.13}$$

Therefore, combining (2.10)–(2.13) as $n\to\infty$ (or, equivalently, as $N\to\infty$), we obtain

$$\begin{aligned} &\ln g^{(d)}_{r,s:n}(\xi,\eta)= \\ &= r\ln\left(1-\frac{\xi\tau_r}{\lambda_r}\right)+(s-r)\ln\left(1-\frac{\eta\tau_s-\xi\tau_r}{\lambda_s-\lambda_r}\right)+(N-s)\ln\left(1+\frac{\eta\tau_s}{\nu_s}\right) \\ &\sim -\frac{\xi^2\nu_r}{2}-\frac{\eta^2\tau_s^2-2\xi\eta\tau_r\tau_s+\xi^2\tau_r^2}{2(\lambda_s-\lambda_r)}(N+1)-\frac{1}{2}\eta^2\lambda_s \\ &= -\frac{\xi^2\nu_r}{2}\left(1+\frac{\lambda_r}{\lambda_s-\lambda_r}\right)-\frac{1}{2}\eta^2\lambda_s\left(1+\frac{\nu_s}{\lambda_s-\lambda_r}\right)-\frac{1}{2}\left(-2\xi\eta\frac{\tau_r\tau_s}{\lambda_s-\lambda_r}\right) \\ &= -\frac{1}{2}\frac{\lambda_s(1-\lambda_r)}{\lambda_s-\lambda_r}\left(\xi^2+\eta^2-2\xi\eta\sqrt{\frac{\lambda_r(1-\lambda_s)}{\lambda_s(1-\lambda_r)}}\right), \end{aligned}$$

which implies

$$g^{(d)}_{r,s:n}(\xi,\eta)=\exp\left(-\frac{\xi^2+\eta^2-2\xi\eta R_{rs}}{2(1-R^2_{rs})}\right)\big(1+o(1)\big).$$

Therefore, for sufficiently large n (or, equivalently, for large N), we obtain

$$\Big|\int\limits_{1-F^{m+1}(\tilde{y}_n)}^{\lambda_r+\tau_r T}\int\limits_{1-F^{m+1}(\tilde{x}_n)}^{\lambda_s+\tau_s T}\varphi^{d(m,k)}_{r,s:n}(w,z)dzdw-\int\limits_{-T}^{U^{d(1)}_n(y)}\int\limits_{-T}^{U^{d(2)}_n(x)}W_{r,s}(\xi,\eta)d\xi d\eta\Big|<\epsilon.$$

Since

$$\int\limits_{-\infty}^{-T}\int\limits_{-T}^{U^{d(2)}_n(x)}W_{r,s}(\xi,\eta)d\xi d\eta+\int\limits_{-\infty}^{U^{d(1)}_n(y)}\int\limits_{-\infty}^{-T}W_{r,s}(\xi,\eta)d\xi d\eta<2\mathcal{N}(-T)<2\epsilon$$

and

$$\int\limits_{-\infty}^{U^{d(1)}_n(y)}\int\limits_{-\infty}^{U^{d(2)}_n(x)}W_{r,s}(\xi,\eta)d\xi d\eta=\int\limits_{-\infty}^{U^{d(1)}_n(y)}\int\limits_{-\infty}^{-T}W_{r,s}(\xi,\eta)d\xi d\eta$$
$$+\int\limits_{-\infty}^{-T}\int\limits_{-T}^{U^{d(2)}_n(x)}W_{r,s}(\xi,\eta)d\xi d\eta+\int\limits_{-T}^{U^{d(1)}_n(y)}\int\limits_{-T}^{U^{d(2)}_n(x)}W_{r,s}(\xi,\eta)d\xi d\eta,$$

we have

$$\Big|\int\limits_{1-F^{m+1}(\tilde{y}_n)}^{\lambda_r+\tau_r T}\int\limits_{1-F^{m+1}(\tilde{x}_n)}^{\lambda_s+\tau_s T}\varphi^{d(m,k)}_{r,s:n}(w,z)dzdw-\int\limits_{-\infty}^{U^{d(1)}_n(y)}\int\limits_{-\infty}^{U^{d(2)}_n(x)}W_{r,s}(\xi,\eta)d\xi d\eta\Big|<3\epsilon.$$

By combining the last inequality with (2.8) and (2.9) we get for sufficiently large n

$$\Big|\Phi^{d(m,k)}_{r,s:n}(\tilde{x}_n,\tilde{y}_n)-\int\limits_{-\infty}^{U^{d(1)}_n(y)}\int\limits_{-\infty}^{U^{d(2)}_n(x)}W_{r,s}(\xi,\eta)d\xi d\eta\Big|<5\epsilon,$$

which proves the lemma in the case $|U^{d(1)}_n(y)|,|U^{d(2)}_n(x)|<T$. This completes the proof. ■

Lemma 2.2 yields directly the following interesting theorem, which characterizes the possible limit laws of the df of the bivariate central m-dgos.

THEOREM 2.1. *The convergence of two marginals*

$$\Phi^{d(m,k)}_{r:n}(\tilde{y}_n)\xrightarrow[n]{w}\Phi^{d(m,k)}(y;\lambda_1)=\mathcal{N}\big(\tilde{U}(y)\big)$$

and

$$\Phi^{d(m,k)}_{s:n}(\tilde{x}_n) \xrightarrow[n]{w} \Phi^{d(m,k)}(x;\lambda_2) = \mathcal{N}\big(U(x)\big),$$

where $\mathcal{N}\big(U(x)\big)$ *and* $\mathcal{N}\big(\tilde{U}(y)\big)$ *are nondegenerate df's, is a necessary and sufficient condition for the convergence of the joint df* $\Phi^{d(m,k)}_{r,s:n}(\tilde{x}_n,\tilde{y}_n)$ *to the nondegenerate limit*

$$\Phi^{d(m,k)}(x,y;\lambda_1,\lambda_2) = \frac{1}{2\pi\sqrt{1-R^2}} \int\limits_{-\infty}^{\tilde{U}(y)} \int\limits_{-\infty}^{U(x)} \exp\left(-\frac{\xi^2+\eta^2-2\xi\eta R}{2(1-R^2)}\right) d\xi d\eta,$$

where $R=\sqrt{\lambda_1(1-\lambda_2)/\big(\lambda_2(1-\lambda_1)\big)}$. *Moreover, the convergence of the bivariate df* $\Phi^{d(m,k)}_{r,s:n}(\tilde{x}_n,\tilde{y}_n)$, *as well as the convergence of two marginals* $\Phi^{d(m,k)}_{r:n}(\tilde{y}_n)$ *and* $\Phi^{d(m,k)}_{s:n}(\tilde{x}_n)$, *occurs if and only if*

$$\Phi^{d(0,1)}_{r:n}(\tilde{y}_n) \xrightarrow[n]{w} \mathcal{N}\big(W_{i;\beta}(y)\big) \quad \text{and} \quad \Phi^{d(0,1)}_{s:n}(\tilde{x}_n) \xrightarrow[n]{w} \mathcal{N}\big(W_{j;\beta'}(x)\big)$$

for some $i,j\in\{1,2,3,4\}$, *where* $\lambda_t(m) = 1-\overline{\lambda}_t^{1/(m+1)}$, $\overline{\lambda}_t = 1-\lambda_t,\ t=1,2$. *In this case we have*

$$\tilde{U}(y) = \frac{C^*_{\lambda_1(m)}}{C^*_{\lambda_1}}(m+1)W_{i;\beta}(y) \quad \text{and} \quad U(x) = \frac{C^*_{\lambda_2(m)}}{C^*_{\lambda_2}}(m+1)W_{j;\beta'}(x),$$

where $C^*_{\lambda_t} = C_{\lambda_t}/\overline{\lambda},\ t=1,2.$

3. LIMIT DISTRIBUTION FUNCTIONS OF THE BIVARIATE INTERMEDIATE m-dgos

A wide class of intermediate oos where $r=r_n=\ell^2 n^\alpha\big(1+o(1)\big),\ 0<\alpha<1$, was studied by Chibisov [5] who showed that if there are normalizing constants $\tilde{\alpha}_n>0$ and $\tilde{\beta}_n$ such that

$$\Phi^{d(0,1)}_{r:n}(\tilde{\alpha}_n x+\tilde{\beta}_n) = \mathbf{I}_{F(\tilde{\alpha}_n x+\tilde{\beta}_n)}(n-r+1,r) \xrightarrow[n]{w} \Phi^{d(0,1)}(x), \tag{3.1}$$

where $\Phi^{d(0,1)}(x)$ is a nondegenerate df, then $\Phi^{d(0,1)}(x)$ must have one and only one of the types $\mathcal{N}\big(V_i(x)\big),\ i=1,2,3$, where $V_1(x)=x$ for all x, and

$$V_2(x) = \begin{cases} -\beta\ln|x|, & x\leqslant 0,\\ \infty, & x>0,\end{cases} \qquad V_3(x) = \begin{cases} -\infty, & x\leqslant 0,\\ \beta\ln|x|, & x>0,\end{cases} \tag{3.2}$$

where β is some positive constant. In this case we say that F belongs to the *domain of attraction* of the df $\Phi^{d(0,1)}(x)$, written $F\in\mathcal{D}\big(\Phi^{d(0,1)}(x)\big)$. Moreover, (3.1) is satisfied with $\Phi^{d(0,1)}(x)=\mathcal{N}\big(V_i(x)\big)$ for some $i\in\{1,2,3\}$ if and only if

$$\frac{r-n\overline{F}(\tilde{\alpha}_n x+\tilde{\beta}_n)}{\sqrt{r}} \xrightarrow[n]{} V_i(x). \tag{3.3}$$

Wu [14] generalized the Chibisov result for any nondecreasing intermediate rank sequence and proved that the only possible types for the limit df of the intermediate oos are those defined in (3.2).

Barakat [1] in Lemma 2.2 and Theorem 2.3 characterized the possible limit laws of the df of the upper intermediate m-gos. The following corresponding lemma characterizes the possible limit laws of the df of the lower intermediate m-dgos.

LEMMA 3.1. *Let $m_1 = m_2 = \ldots = m_{n-1} = m > -1$, and let r_n be a nondecreasing intermediate rank sequence. Then there exist normalizing constants $\tilde{a}_n > 0$ and $\tilde{b}_n$ such that*

$$\Phi^{d(m,k)}_{r_n:n}(\tilde{a}_n x + \tilde{b}_n) \xrightarrow[n]{w} \Phi^{d(m,k)}(x), \tag{3.4}$$

where $\Phi^{d(m,k)}(x)$ is a nondegenerate df if and only if

$$\frac{r_N - N\overline{T}_m(\tilde{a}_n x + \tilde{b}_n)}{\sqrt{r_N}} \xrightarrow[n]{} V(x),$$

where $\Phi^{d(m,k)}(x) = \mathcal{N}\big(V(x)\big)$. Furthermore, let r_n^* be a variable rank sequence defined by $r_n^* = r_{\theta^{-1}(N)}$, with $\theta(n) = (m+1)N$ (remember that $N = k/(m+1) + n - 1$; then $\theta(n) = n$ if $m = 0, k = 1$, i.e., in the case of oos). Then there exist normalizing constants $\tilde{a}_n > 0$ and $\tilde{b}_n$ for which (3.4) is satisfied for some nondegenerate df $\Phi^{d(m,k)}(x)$ if and only if there are normalizing constants $\tilde{\alpha}_n > 0$ and $\tilde{\beta}_n$ for which $\Phi^{d(0,1)}_{r_n^*:n}(\tilde{\alpha}_n x + \tilde{\beta}_n) \xrightarrow[n]{w} \Phi^{d(0,1)}(x)$, where $\Phi^{d(0,1)}(x)$ is some nondegenerate df, or, equivalently, $\big(r_n^* - n\overline{F}(\tilde{\alpha}_n x + \tilde{\beta}_n)\big)/\sqrt{r_n^*} \xrightarrow[n]{} V_i(x)$, $i \in \{1,2,3\}$, and $\Phi^{d(0,1)}(x) = \mathcal{N}\big(V_i(x)\big)$. In this case $\tilde{a}_n$ and $\tilde{b}_n$ may be chosen such that $\tilde{a}_n = \tilde{\alpha}_{\theta(n)}$ and $\tilde{b}_n = \tilde{\beta}_{\theta(n)}$. Moreover, $\Phi^{d(m,k)}(x)$ must have the form $\mathcal{N}\big(V_i(x)\big)$, i.e., $V(x) = V_i(x)$.

P r o o f. The proof follows by using the same argument which is applied in the proof of Lemma 2.2 and Theorem 2.3 of Barakat [1] in the case of upper intermediate m-gos. ∎

Now, we consider the limit df of two intermediate m-dgos

$$\eta_r^{(d)} = \frac{X_d(r,n,m,k) - \tilde{b}_n}{\tilde{a}_n} \quad \text{and} \quad \zeta_s^{(d)} = \frac{X_d(s,n,m,k) - \tilde{d}_n}{\tilde{c}_n},$$

where $r/n^{\alpha_1} \xrightarrow[n]{} l_1^2, s/n^{\alpha_2} \xrightarrow[n]{} l_2^2, 0 < \alpha_1, \alpha_2 < 1, l_1, l_2 > 0$, and $\tilde{a}_n, \tilde{c}_n > 0, \tilde{b}_n, \tilde{d}_n$ are suitable normalizing constants. Our main aim is:

1. to prove that the weak convergence of the df's of $\eta_r^{(d)}$ and $\zeta_s^{(d)}$ implies the convergence of the bivariate df of $\eta_r^{(d)}$ and $\zeta_s^{(d)}$;

2. to obtain the limit joint df of $\eta_r^{(d)}$ and $\zeta_s^{(d)}$ and derive the condition under which the two statistics $\eta_r^{(d)}$ and $\zeta_s^{(d)}$ are asymptotically independent.

We can distinguish the following distinct and exhausted two cases:

(A) $s-r \underset{n}{\to} c,\ 0 \leqslant c < \infty$.

(B) $s-r \underset{n}{\to} \infty$.

REMARK 3.1. *Under the condition* (A) *we clearly have* $l_1 = l_2, \alpha_1 = \alpha_2 = \alpha$. *Moreover*, $r/s \underset{n}{\to} 1$. *Finally*, *under the condition* (B) *we have the following three distinct and exhausted cases*:

(B$_1$) $\alpha_2 > \alpha_1$, *which implies* $r/s \underset{n}{\to} 0$.

(B$_2$) $\alpha_2 = \alpha_1 = \alpha, l_2 > l_1$, *which implies* $r/s \underset{n}{\to} l_1^2/l_2^2$.

(B$_3$) $\alpha_2 = \alpha_1 = \alpha, l_2 = l_1$, *which implies* $r/s \underset{n}{\to} 1$.

The following lemma, corresponding to Lemma 2.2, characterizes the possible limit laws of the bivariate intermediate m-dgos.

LEMMA 3.2. *Let us assume that* $\Phi_{r,s:n}^{d(m,k)}(\tilde{x}_n, \tilde{y}_n) = P(\eta_r^{(d)} < x, \zeta_s^{(d)} < y), 0 < R_{rs} = \sqrt{\lambda_r(1-\lambda_s)/\big(\lambda_s(1-\lambda_r)\big)} < 1$, $\tilde{x}_n = \tilde{a}_n x + \tilde{b}_n, \tilde{y}_n = \tilde{c}_n y + \tilde{d}_n$, *and let*

$$U_n^{d(1)}(y) = \frac{\lambda_r - \overline{T}_m(\tilde{y}_n)}{\tau_r}, \qquad U_n^{d(2)}(x) = \frac{\lambda_s - \overline{T}_m(\tilde{x}_n)}{\tau_s},$$

$\lambda_i = i/(N+1), \tau_i = \sqrt{\lambda_i \nu_i/(N+1)}$ *and* $\nu_i = 1 - \lambda_i,\ i = r, s$. *Then*

$$\Bigg| \Phi_{r,s:n}^{d(m,k)}(\tilde{x}_n, \tilde{y}_n) - \frac{1}{2\pi\sqrt{1-R_{rs}^2}} \times \int_{-\infty}^{U_n^{d(1)}(y)} \int_{-\infty}^{U_n^{d(2)}(x)} \exp\left(-\frac{\xi^2 + \eta^2 - 2\xi\eta R_{rs}}{2(1-R_{rs}^2)}\right) d\xi d\eta \Bigg| \underset{n}{\to} 0$$

uniformly with respect to x *and* y.

Proof. The proof is very close to the proof of Lemma 2.2. Therefore, we show only the necessary changes in the proof of Lemma 2.2. First, we begin the proof, as we have done in Lemma 2.2, by choosing T, for given $\epsilon > 0$, large enough to satisfy both of the inequalities $1/T^2 < \epsilon$ and $\mathcal{N}(-T) < \epsilon$. In this case it is easy to see that the proof of the two lemmas coincides in the cases $U_n^{d(t)}(\cdot) \leqslant -T$ and $U_n^{d(t)}(\cdot) \geqslant T,\ t = 1, 2$. Therefore, we will prove the lemma only in the case $|U_n^{d(1)}(y)| < T$ and $|U_n^{d(2)}(x)| < T$. In this case we have $1 - F^{m+1}(\tilde{y}_n) < \lambda_r +$

$\tau_r T \geqslant 0$ and $1 - F^{m+1}(\tilde{x}_n) < \lambda_s + \tau_s T \geqslant 0$. Thus, we get

$$\text{(3.5)} \quad \Phi_{r,s:n}^{d(m,k)}(\tilde{x}_n, \tilde{y}_n) = \int\limits_{1-F^{m+1}(\tilde{y}_n)}^{\lambda_r+\tau_r T} \int\limits_{1-F^{m+1}(\tilde{x}_n)}^{\lambda_s+\tau_s T} \varphi_{r,s:n}^{d(m,k)}(w,z)dzdw$$

$$+ \int\limits_{1-F^{m+1}(\tilde{y}_n)}^{\lambda_r+\tau_r T} \int\limits_{\lambda_s+\tau_s T}^{w} \varphi_{r,s:n}^{d(m,k)}(w,z)dzdw + \int\limits_{\lambda_r+\tau_r T}^{1} \int\limits_{1-F^{m+1}(\tilde{x}_n)}^{w} \varphi_{r,s:n}^{d(m,k)}(w,z)dzdw,$$

where

$$\varphi_{r,s:n}^{(m,k)}(w,z) = \frac{C_n^*}{(m+1)^2} z^{r-1}(1-w)^{N-s}(w-z)^{s-r-1}.$$

We shall separately consider each of the integrals in the summation (3.5). We have

$$\int\limits_{\lambda_r+\tau_r T}^{1} \int\limits_{1-F^{m+1}(\tilde{x}_n)}^{w} \varphi_{r,s:n}^{d(m,k)}(w,z)dzdw \leqslant \int\limits_{0}^{\lambda_r+\tau_r T} \int\limits_{1}^{w} \varphi_{r,s:n}^{d(m,k)}(w,z)dzdw$$

$$= \frac{\Gamma(N+1)}{\Gamma(N-r+1)(r-1)!} \int\limits_{0}^{\lambda_r+\tau_r T} w^{r-1}(1-w)^{N-r}dw = \frac{N+1}{(N+2)T^2} < \epsilon.$$

Since $|U_n^{d(1)}(y)| < T$, for large N we get

$$\text{(3.6)} \qquad 1 - F^{m+1}(\tilde{y}_n) < \lambda_r + \tau_r T.$$

On the other hand, we have

$$\text{(3.7)} \qquad \frac{\lambda_r + \tau_r T}{\lambda_s - \tau_s T} \underset{n}{\rightarrow} \begin{cases} 0 & \text{in the case } (\mathrm{B}_1), \\ l_l^2/l_2^2 & \text{in the case } (\mathrm{B}_2), \\ 1 & \text{in the cases (A) and } (\mathrm{B}_3). \end{cases}$$

Therefore, for large N, the relations (3.6) and (3.7) imply the inequality

$$\text{(3.8)} \qquad 1 - F^{m+1}(\tilde{y}_n) < \lambda_s + \tau_s T.$$

The inequality (3.8) leads to the following estimate for the second integral in (3.5):

$$\int\limits_{1-F^{m+1}(\tilde{y}_n)}^{\lambda_r+\tau_r T} \int\limits_{\lambda_s+\tau_s T}^{w} \varphi_{r,s:n}^{d(m,k)}(w,z)dzdw \leqslant \int\limits_{0}^{\lambda_s+\tau_s T} \int\limits_{w}^{\lambda_s+\tau_s T} \varphi_{r,s:n}^{d(m,k)}(w,z)dzdw$$

$$= \int\limits_{0}^{\lambda_s+\tau_s T} \int\limits_{0}^{z} \varphi_{r,s:n}^{d(m,k)}(w,z)dwdz = \frac{\Gamma(N+1)}{\Gamma(N-s+1)(s-1)!}$$

$$\times \int\limits_{0}^{\lambda_s+\tau_s T} z^{s-1}(1-z)^{N-s}dz < \frac{1}{T^2} < \epsilon.$$

It is easy to show that under the conditions (B_1) and (B_2) the mathematical treatments of the third integral of the summation, as well as the remaining part of the proof, is exactly the same as in the proof of Lemma 2.2. Consequently, we consider only the third integral under the conditions (A) and (B_3). It is convenient now to divide the case (B_3) into the following two cases:

(B_{3a}) $s - r = o(N^\alpha) = a\, N^\beta + o(N^\beta),\ a > 0,\ \beta \leqslant \alpha/2;$

(B_{3b}) $s - r = o(N^\alpha) = a\, N^\beta + o(N^\beta),\ a > 0,\ \beta > \alpha/2.$

Now, in view of (3.8) we get, after some simple calculations,

$$\begin{aligned}
\int\limits_{1-F^{m+1}(\tilde{y}_n)}^{\lambda_r+\tau_r T}\int\limits_{1-F^{m+1}(\tilde{x}_n)}^{\lambda_s+\tau_s T} & \varphi^{d(m,k)}_{r,s:n}(w,z)dzdw \\
&\leqslant \int\limits_0^{\lambda_s+\tau_s T}\int\limits_z^{\lambda_s+\tau_s T}\varphi^{d(m,k)}_{r,s:n}(w,z)dwdz \\
&\leqslant \int\limits_0^{\lambda_s+\tau_s T}\int\limits_0^{w}\varphi^{d(m,k)}_{r,s:n}(w,z)dzdw \\
&= \frac{\Gamma(N+1)}{\Gamma(N-r+1)\Gamma(r)}\int\limits_0^{\lambda_s+\tau_s T} w^{r-1}(1-w)^{N-r}dw \\
&\leqslant \frac{\Gamma(N+1)}{\Gamma(N-r+1)\Gamma(r)}\int\limits_0^{1}\frac{(w-\lambda_s)^2}{\tau_s^2T^2}w^{r-1}(1-w)^{N-r}dw \\
&= \frac{1}{\tau_s^2T^2}\left\{\frac{r(r+1)}{(N+1)(N+2)}+\lambda_s^2-\frac{2r\lambda_s}{N+1}\right\} \\
&= \frac{1}{T^2}\left\{\frac{(s-r)^2}{s(1-\lambda_s)}+\frac{(N+1)}{(N+2)}\frac{(1-\lambda_r)}{(1-\lambda_s)}\frac{r}{s}\right\}.
\end{aligned}$$

Since

$$\frac{(N+1)}{(N+2)}\frac{(1-\lambda_r)}{(1-\lambda_s)}\frac{r}{s} \underset{n}{\to} 1$$

and

$$\frac{(s-r)^2}{s(1-\lambda_s)} \underset{n}{\to} \begin{cases} 0 & \text{in the case (A)}, \\ 0 & \text{in the case } (\mathrm{B}_{3a}), \beta < \alpha/2, \\ a^2/l^2 & \text{in the case } (\mathrm{B}_{3a}), \beta = \alpha/2, \end{cases}$$

we get

$$\int\limits_{1-F^{m+1}(\tilde{y}_n)}^{\lambda_r+\tau_r T}\int\limits_{1-F^{m+1}(\tilde{x}_n)}^{\lambda_s+\tau_s T}\varphi^{d(m,k)}_{r,s:n}(w,z)dzdw < \epsilon\left(1+\frac{a^2}{l^2}+2\epsilon\right).$$

On the other hand, since under the conditions (A) and (B_{3a}) we have $R_{rs} \underset{n}{\to} 1$, it follows that

$$\int_{-T}^{U_n^{d(1)}(y)} \int_{-T}^{U_n^{d(2)}(x)} W_{r,s}(\xi,\eta) d\xi d\eta$$

$$\leqslant \frac{1}{2\pi\sqrt{1-R_{rs}^2}} \int_{-T}^{T}\int_{-T}^{T} \exp\left(-\frac{\xi^2+\eta^2-2\xi\eta R_{rs}}{2(1-R_{rs}^2)}\right) d\xi d\eta \underset{n}{\to} 0.$$

Therefore, we get

$$\left|\Phi_{r,s:n}^{d(m,k)}(\tilde{x}_n,\tilde{y}_n) - \int_{-\infty}^{U_n^{d(1)}(y)} \int_{-\infty}^{U_n^{d(2)}(x)} W_{r,s}(\xi,\eta) d\xi d\eta\right| < \epsilon\left(6+\frac{a^2}{l^2}+2\epsilon\right),$$

which completes the proof of the lemma in the cases (A) and (B_{3a}). It remains to prove the case (B_{3b}). By using the transformation $z = \lambda_r - \xi\tau_r$, $w = \lambda_s - \eta\tau_s$, the third integral takes the form

$$\int_{1-F^{m+1}(\tilde{y}_n)}^{\lambda_r+\tau_r T} \int_{1-F^{m+1}(\tilde{x}_n)}^{\lambda_s+\tau_s T} \varphi_{r,s:n}^{d(m,k)}(w,z) dz dw = A_{r,s:n} \int_{-T}^{U_n^{d(1)}(y)} \int_{-T}^{U_n^{d(2)}(x)} g_{r,s:n}^{(d)}(\xi,\eta) d\eta d\xi,$$

where

$$A_{r,s:n} = \frac{\Gamma(N+1)\tau_r\tau_s\lambda_r^{r-1}\nu_s^{N-s}(\lambda_s-\lambda_r)^{s-r-1}}{\Gamma(N-s+1)(r-1)!(s-r-1)!}$$

and

$$g_{r,s:n}^{(d)}(\xi,\eta) = \left(1-\frac{\xi\tau_r}{\lambda_r}\right)^{r-1}\left(1-\frac{\eta\tau_s-\xi\tau_r}{\lambda_s-\lambda_r}\right)^{s-r-1}\left(1+\frac{\eta\tau_s}{\nu_s}\right)^{N-s}. \tag{3.9}$$

Therefore, using Stirling's formula we get $A_{r,s:n} = \big(1+o(1)\big)/(2\pi\sqrt{1-R_{rs}^2})$. Furthermore,

$$g_{r,s:n}^{(d)}(\xi,\eta) = \left(1-\frac{\xi\tau_r}{\lambda_r}\right)^{r}\left(1-\frac{\eta\tau_s-\xi\tau_r}{\lambda_s-\lambda_r}\right)^{s-r}\left(1+\frac{\eta\tau_s}{\nu_s}\right)^{N-s}\big(1+\rho_n(\xi,\eta)\big),$$

where $\rho_n(\xi,\eta) \underset{n}{\to} 0$. On the other hand, it can be shown that

$$\frac{\eta\tau_s-\xi\tau_r}{\lambda_s-\lambda_r} = \frac{\eta\sqrt{\lambda_s(1-\lambda_s)/(N+1)}}{(s-r)/(N+1)} - \frac{\xi\sqrt{\lambda_r(1-\lambda_r)/(N+1)}}{(s-r)/(N+1)}$$

$$= \frac{s^{1/2}}{s-r}\eta(1-\lambda_s)^{1/2} - \frac{r^{1/2}}{s-r}\xi(1-\lambda_r)^{1/2} \underset{n}{\to} 0,$$

$$r\ln\left(1-\frac{\xi\tau_r}{\lambda_r}\right)=-\xi\tau_r(N+1)-\frac{\xi^2\nu_r}{2}+o\left(\frac{T^3}{\sqrt{r}}\right), \tag{3.10}$$

$$\begin{aligned}(s-r)\ln\left(1-\frac{\eta\tau_s+\xi\tau_r}{\lambda_s-\lambda_r}\right) &= -(s-r)\left[\frac{\eta\tau_s-\xi\tau_r}{\lambda_s-\lambda_r}+\frac{1}{2}\left(\frac{\eta\tau_s-\xi\tau_r}{\lambda_s-\lambda_r}\right)^2+\frac{1}{3}\left(\frac{\eta\tau_s-\xi\tau_r}{\lambda_s-\lambda_r}\right)^3+\dots\right]\\ &= -(\eta\tau_s-\xi\tau_r)(N+1)-\frac{1}{2}\frac{(\eta\tau_s-\xi\tau_r)^2}{\lambda_s-\lambda_r}(N+1)\\ &\quad -\frac{(\eta-\xi)^3l^3}{3a^2}N^{(3\alpha)/2-2\beta}\big(1+o(1)\big)\end{aligned} \tag{3.11}$$

and

$$(N-s)\ln\left(1+\frac{\eta\tau_s}{\nu_s}\right)=\eta\tau_s(N+1)-\frac{1}{2}\eta^2\lambda_s+o\left(\frac{\lambda_s^{3/2}T^3}{\sqrt{N}}\right). \tag{3.12}$$

Therefore, combining the relations (3.9)–(3.12), we get

$$\begin{aligned}g^{(d)}_{r,s:n}(\xi,\eta)&=\exp\left(-\frac{\xi^2+\eta^2-2\xi\eta R_{rs}}{2(1-R^2_{rs})}\right.\\ &\quad\times\left.\left(1+\frac{2(\xi-\eta)^3l}{3a(\xi^2+\eta^2-2\xi\eta R_{rs})}N^{\alpha/2-\beta}\big(1+o(1)\big)\right)\right)\big(1+o(1)\big)\underset{n}{\to}0\end{aligned}$$

since $R_{rs}\underset{n}{\to}1$. Thus, we have

$$\int\limits_{1-F^{m+1}(\tilde{y}_n)}^{\lambda_r+\tau_rT}\int\limits_{1-F^{m+1}(\tilde{x}_n)}^{\lambda_s+\tau_sT}\varphi^{d(m,k)}_{r,s:n}(w,z)dzdw\underset{n}{\to}0. \tag{3.13}$$

On the other hand, we obtain

$$\int\limits_{1-F^{m+1}(\tilde{y}_n)}^{\lambda_r+\tau_rT}\int\limits_{1-F^{m+1}(\tilde{x}_n)}^{\lambda_s+\tau_sT}\varphi^{d(m,k)}_{r,s:n}(w,z)dzdw<\int\limits_{-T}^{T}\int\limits_{-T}^{T}W_{r,s}(\xi,\eta)d\xi d\eta\underset{n}{\to}0 \tag{3.14}$$

since $R_{rs}\underset{n}{\to}1$. The relations (3.13) and (3.14) prove that the third integral in (3.5) converges to zero. This completes the proof of Lemma 3.2. ∎

The following theorem characterizes the possible limit laws of the df of the bivariate intermediate m-dgos.

THEOREM 3.1. *Let* $r/n, s/n\underset{n}{\to}0, r/s\underset{n}{\to}R$, *and* $R_{rs}\underset{n}{\to}\sqrt{R}$, $0\leqslant R\leqslant 1$. *Then the convergence of two marginals* $\Phi^{d(m,k)}_{r:n}(\tilde{y}_n)\overset{w}{\underset{n}{\to}}\Phi^{d(m,k)}(y)=\mathcal{N}\big(\tilde{H}(y)\big)$

and $\Phi^{d(m,k)}_{s:n}(\tilde{x}_n) \xrightarrow[n]{w} \Phi^{d(m,k)}(x) = \mathcal{N}\big(H(x)\big)$, *where* $\mathcal{N}\big(H(x)\big)$ *and* $\mathcal{N}\big(\tilde{H}(y)\big)$ *are nondegenerate df's, is a necessary and sufficient condition for the convergence of the joint df* $\Phi^{d(m,k)}_{r,s:n}(\tilde{x}_n, \tilde{y}_n)$ *to the nondegenerate limit*

$$\Phi^{d(m,k)}_{r,s:n}(\tilde{x}_n, \tilde{y}_n) \xrightarrow[n]{w} \frac{1}{2\pi\sqrt{1-R}} \int\limits_{-\infty}^{\tilde{H}(y)} \int\limits_{-\infty}^{H(x)} \exp\left(-\frac{\xi^2+\eta^2-2\xi\eta\sqrt{R}}{2(1-R)}\right) d\xi d\eta.$$

Moreover, *the convergence of the bivariate df* $\Phi^{d(m,k)}_{r,s:n}(\tilde{x}_n, \tilde{y}_n)$, *as well as the convergence of two marginals* $\Phi^{d(m,k)}_{r:n}(\tilde{y}_n)$ *and* $\Phi^{d(m,k)}_{s:n}(\tilde{x}_n)$, *occurs if and only if there are normalizing constants* $\tilde{\alpha}_n, \tilde{\gamma}_n > 0$ *and* $\tilde{\beta}_n, \tilde{\delta}_n$ *for which*

$$\Phi^{d(0,1)}_{r^*_n:n}(\tilde{\alpha}_n y + \tilde{\beta}_n)(= \Phi^{(0,1)}_{n-r^*_n+1:n}(\tilde{\alpha}_n y + \tilde{\beta}_n)) \xrightarrow[n]{w} \Phi^{d(0,1)}(y) = \mathcal{N}\big(V_j(y)\big)$$

and

$$\Phi^{d(0,1)}_{s^*_n:n}(\tilde{\gamma}_n x + \tilde{\delta}_n) \xrightarrow[n]{w} \Phi^{d(0,1)}(x) = \mathcal{N}\big(V_i(x)\big)$$

for some $i, j \in \{1,2,3\}$, *where* $r^*_n = r_{\theta^{-1}(N)}$, $s^*_n = s_{\theta^{-1}(N)}$, *and* $\theta(n) = (m+1)N$. *In this case*, *we can take* $\tilde{a}_n = \tilde{\alpha}_{\theta(n)}, \tilde{c}_n = \tilde{\gamma}_{\theta(n)}, \tilde{b}_n = \tilde{\beta}_{\theta(n)}$ *and* $\tilde{d}_n = \tilde{\delta}_{\theta(n)}$. *Moreover*, $H(x) = V_i(x)$ *and* $\tilde{H}(y) = V_j(y)$. *Finally*, *the two marginals are asymptotically independent if and only if* $r/s \xrightarrow[n]{} 0$, *i.e.*, $R = 0$.

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